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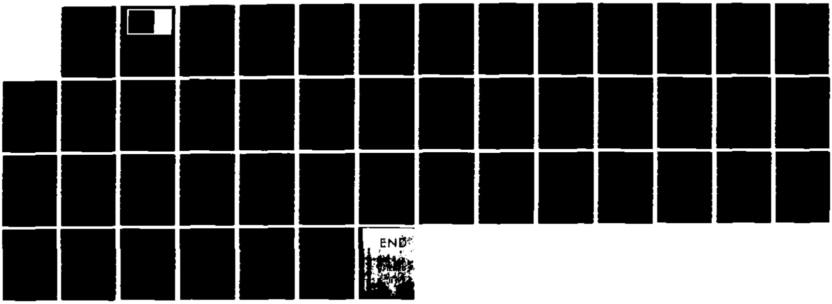

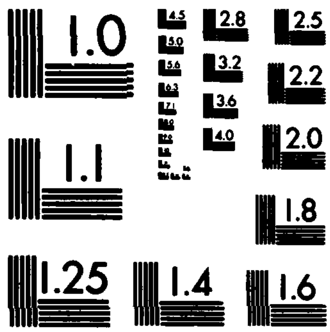

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MRC Technical Summary Report #2544

KEYNESIAN CHAOS

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UNIVERSITY OF WISCONSIN-MADISON
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ABSTRACT

This paper shows how chaotic fluctuations can emerge in a dynamic version of the standard fixprice macroeconomic model. The crucial parameters involve induced investment and its sensitivity to interest rates. We use some ideas from the theory of chaos and from ergodic theory to illustrate the fact that apparently random trajectories need not be rare. The paper concludes with some tantalizing hints at how insights on monetary theory and policy might change when comparative dynamics rather than comparative statics are used and when nonlinearities of the kind illustrated in the paper prevail.

AMS (MOS) Subject Classifications: 90A16, 34C35, 39A10, 28D99

Key Words: Chaotic Dynamics, Ergodic Behavior, Nonlinear Dynamics, Keynesian Economics

Work Unit Numbers 6 and 1 (Miscellaneous Topics and Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

Chaotic behavior is a term used to describe solutions of difference or differential equations that wander in a bounded, uncountable set without converging to a periodic trajectory. Chaotic trajectories have the further properties of wandering close to each other and away from each other, and away from any periodic trajectory which they may approach.

The existence of such solutions for models of empirical phenomona provides a theoretical explanation of the irregular or erratic character often observed in real world data.

If a model is shown to be "ergodic" it means that it behaves qualitatively much like a stochastic process even though it is "deterministic", that is, even though no random variables are part of the model assumptions.

This paper shows how chaotic and ergodic behavior can emerge in the standard macroeconomic model of Keynes. It suggests the possibility that the extreme difficulties in forecasting economic variables may be due to intrinsic forces rather than or in addition to shocks that perturb what would otherwise be a stable system.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

KEYNESIAN CHAOS

by

Richard H. Day and Wayne Shafer

> ...but regular fluctuation, on this pattern, has not persisted. The economic world, it has in our day become increasingly obvious, is inherently in a state of flux.
>
> Sir John Hicks in *Causality in Economics*

1. THEORY K: NONLINEAR SCRAMBLING

The persistent irregularity exhibited by macroeconomic variables such as GNP, interest rates, unemployment rates, etc., is usually attributed to random shocks.[1] Recently, Sargent, for example, observed that

> ...if the inital conditions of low-order, deterministic linear difference equations are subjected to repeated random shocks of a certain kind, there emerges the possibility of recurring, somewhat irregular cycles of the kind seemingly infesting economic data. (Sargent, pp 218-219)

Let us call this, "Theory X."

A contrasting alternative explanation, which was illustrated in two recent articles on irregular growth cycles [Day, 1982, 1983] is based on what might be called, "dynamic scrambling," a phenomenon caused by nonlinearity in deterministic relationships. The possibility that

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nonlinear, deterministic processes can display extremely complicated behavior was recognized early in this century by Poincaré. (Moser, 1973, Ch. 1). Modern work leading to a deeper understanding of such possibilities eminates from the seminal work of Lorenz (1962, 1963a,b, 1964) and Smale (1964). Since then many authors in a variety of fields have contributed to the subject.[2] In all of these nonperiodicity, irregularity, erraticness, chaos, turbulence, or randomness, as the phenomenon is variously called, emerges from the underlying structure of relationships. Let us call this, "Theory K.".

One way to investigate the applicability of Theory K to economics is to see how intrinsic irregularity emerges in various existing models that have already been developed for one purpose or another. This approach indicates that one is not merely applying to economics in an ad hoc fashion ideas developed elsewhere. Instead, it shows that Theory K arises naturally within our subject matter once one knows what to look for. The present paper is another example of this approach.[3]

What we are going to do first is show how chaotic behavior arises in a general fixprice macro model with Robertsonian Lag when induced investment is strong enough.[4] A special case of this general model is next considered in which induced investment can be driven to zero. This leads to a kinked aggregate demand function for which the sufficient chaos conditions can be constructed in a simplified manner. We then take up the Keynesian assumptions about the demand for money and goods for which the results for the kinked, fixprice macromodel apply.

A piecewise linear version of the Keynesian model with nonnegativity constraints is now derived. Advantage can be taken of its sim-

plified structure to show intuitively how unstable trajectories converge -- not to cycles of some periodic order, but to an infinitely "layered," disconnected, uncountable scrambled set or "strange attractor." We then examine the ergodic or stochastic nature of chaotic behavior, deriving for a special case a complete characterization of the nonperiodic trajectories. Although this kind of result is restricted at present to the piecewise linear model, we present some computer simulations that suggest that similar properties hold for more general nonlinear examples.

Our paper concludes with a tantalizing indication that in the dynamic fixprice model - in contrast to the standard comparative static analysis - policies that influence the supply of money can lead to a variety of quite different outcomes depending on the initial conditions. Indeed, three distinct monetary regimes are identified, two with stable Keynesian equilibria separated by one with potential chaotic cycles. By shifting the economy from one regime to another monetary policy can drastically alter qualitative behavior as well as the economy's equilibrium position.

Whether or not we can construct theoretical models that offer convincing explanations of real world macro-activity is an open question: Most economists would probably agree that we are as yet quite far from a definitive answer to it. What we know now as a result of exercises like the one undertaken in this paper is that among the empirical phenomena that we can hope to explain by means of analytical theory are stochastic-like fluctuations in economic data. Moreover, we need not expect that exotic assumptions or bizarre model structures will be required and we may uncover important new implications for economic policy.

2. THE DYNAMIC, FIXPRICE MACRO MODEL

The General Model

Consider the standard macromodel with fixed prices. The monetary sector is represented by the demand for money, $M^D = D(i,Y)$, where i is the interest rate and Y is national income measured in fixed prices; the supply of money, $M^S = S(i,Y;M)$, where M is a money supply parameter; and a market clearing equation, $D(i,Y) = S(i,Y;M)$. The latter implicitly defines the LM curve, $i = L(Y;M)$, which gives the market-clearing interest rate as a function of national income and the money supply parameter.

The real sector is represented by an induced consumption function $C = C(i,Y)$, and an induced investment function, $I = \mu I(i,Y)$, where $\mu \geqq 0$ is a parameter measuring the "strength" or "intensity" of induced investment. Autonomous investment and/or consumption expenditure will be denoted by the parameter "A". Substituting the LM function for interest in the consumption and investment functions we obtain respectively the consumption-income (CY) function

$$(1) \qquad C = G(Y;M) := C[L(Y;M),Y] ,$$

and the investment-income (IY) function

$$(2) \qquad I = \mu H(Y;M) := \mu I[L(Y;M),Y] .$$

In the dynamic analysis that follows these are the basic components to be used rather than the familiar IS and LM functions. However, it must

be remembered that the LM function, which expresses temporary money market equilibrium, is subsumed in both CY and IY relationships.

Now introduce the Robertsonian Lag specifying that current consumption and investment demand depend on lagged income.[5] Instead of an equilibrium equation relating aggregate supply and demand we get the Samuelson demand driven adjustment process

$$Y_{t+1} = \theta(Y_t;\mu,M,A) := G(Y_t;M) + \mu H(Y_t;M) + A \,. \tag{3}$$

This is the basic difference equation whose behavior we want to investigate.

Characteristics of Aggregate Demand

First, we need to be precise about the CY and IY functions. For the former consider

> Assumption A: Consumption $G(Y)$ is a continuous, upward sloping function of income bounded below by a linear consumption function βY where $0 < \beta < 1$ and from above by Y. Assume also that $G(0) = 0$.

Assumption A would be met, for example, if $C(i,Y) := \beta Y$, the usual linear case.

For investment we have

> Assumption B: Investment $H(Y)$ may be an initially increasing function of income (though it need not be) but it must eventually decrease and coincide with or approach zero from above as Y increases. We assume that $H(Y) > 0$ for some $Y > 0$, and that $H(\cdot)$ is continuous.

Examples of investment functions conforming to this assumption are numerous. Note that the linear (affine) demand function is included if there is a nonnegativity "kink" added. The reason investment demand must eventually be downward sloping is that, as income rises the transactions demand for money rises. This causes interest rates, in the presence of a fixed money supply, to rise. If induced investment is elastic enough at high interest rates, then investment demand must eventually fall. These are the usual Keynesian assumptions and we shall consider them in detail below.

The Existence of Chaos

Before proceeding to the analysis we recall that "chaos" is defined for the difference equation $x_{t+1} = \theta(x_t)$ where θ is a real-valued, continuous function mapping an interval J into itself. Here $\theta(\cdot)$ is the aggregate demand function. The sufficient condition obtained by Li and Yorke [1975] in the form used in the present paper is the existence of a point, say $x^c \in J$, such that

$$\text{(L-Y)} \qquad \theta^3(x^c) \geq x^c > \theta(x^c) > \theta^2(x^c)$$

where θ^n is the n-th iterate of θ.[6]

Let us now see how the L-Y sufficient conditions for chaos come about with the increasing importance of induced investment. Let autonomous demand A and the supply of money parameter M be fixed and consider how aggregate demand shifts with μ. If investment goes to zero or approaches it closely enough the effect of increasing μ is not only to raise investment demand but to twist it as shown in Diagram (a) of

Figure 1. The result is both to raise and to twist aggregate demand as well. This is illustrated in Diagram (b) of Figure 1.

As this twisting increases, a local maximum and a local minimum separated by the Keynesian equilibrium emerge . These all depend on μ. Let Y_μ^{max} be the local maximum value of $\theta(Y)$, Y_μ^{min} be the local minimum and let Y_μ^k be the stationary state (Keynesian equilibrium). Note that any income above the stationary state Y_μ^k leads to a successive reduction in income. Consequently, if income initially surpasses this level it must fall below it. Consequently, all time paths for national income must eventually lie in the interval $[Y_\mu^{min}, Y_\mu^{max}]$. If these paths are to keep from bumping into or exceeding full employment we must have $Y_\mu^{max} < Y^F$ where Y^F is full employment income. Therefore, for the dynamic process (3) to be interesting this assumption is needed. Formally,

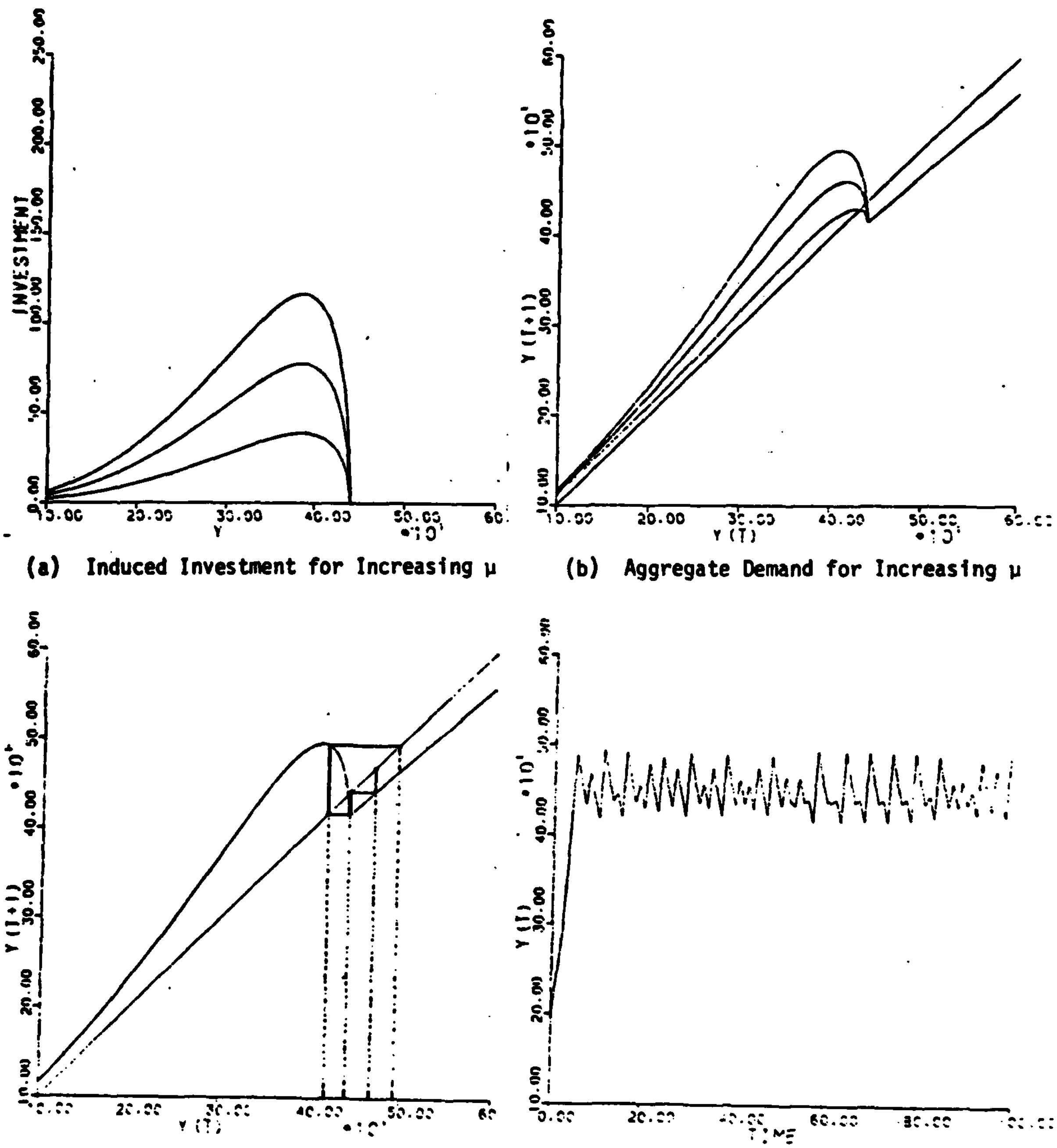
> <u>Assumption C</u>: Assume that H(Y) has the property that for some μ there exists a local finite maximum, say Y_μ^{max} of the aggregate demand curve $\theta_\mu(Y) = G(Y) + \mu H(Y) + A$ such that $Y_\mu^{max} \leq Y^F$.

Now define the preimage of the local minimizer Y_μ^* as that value of Y, say Y_μ^c, <u>that is greater than</u> Y_μ^* and such that $Y_\mu^* = \theta(Y_\mu^c)$.

With these definitions we can state a sufficient condition for chaotic dynamics in the basic fixprice, macro model.

> PROPOSITION I: If there exists CY and IY functions $G(\cdot)$, $H(\cdot)$ satisfying assumption A-C and if in addition there exists μ such that

$$\text{(K)} \qquad \theta(Y_\mu^{min}) \geq Y_\mu^c > Y_\mu^* > Y_\mu^{min}$$



(a) Induced Investment for Increasing μ

(b) Aggregate Demand for Increasing μ

(c) Sufficient Chaos Condition Satisfied

(d) A Sample Chaotic Trajectory

FIGURE 1: Chaos For A "General" Fixprice Model

then there exist cycles of every order and an uncountably infinite number of unstable chaotic trajectories in aggregate income, investment, consumption and employment.

To see that this condition is sufficient just set $Y_0 = Y_\mu^c$. Then $\theta(Y_0) = Y_\mu^*$, $\theta^2(Y_0) = Y_\mu^{min}$ and $\theta^3(Y_0) = \theta(Y_\mu^{min})$ so that Y_0 plays the role of x^c in the L-Y condition given above. Figure 1c shows a point satisfying condition (K) and Figure 1d a segment of a chaotic trajectory for the specific functional form given in Appendix A.

Kinked Investment-Income Functions

To make possible the existence of intensities μ satisfying Condition K, the main requirement is that investment approach zero rapidly enough when interest rates are sufficiently high. This is what gives the twisting effect illustrated in Figure 1. One class of functions that have this property is the one in which investment falls to a constant at some income level Y^o. Simply by adding this constant to autonomous investment and subtracting it from the function I(Y) one obtains an investment schedule for which investment is zero at Y^o. Formally, consider

> Assumption D: There exists a level $Y^o < Y^F$ such that $H(Y) = 0$ all $Y \geq Y^o$.

For models satisfying this assumption (and assumption B) equation (3) has two regimes, one for $Y \varepsilon [0, Y^o]$ and one for $Y \varepsilon [Y^o, Y^F]$. Thus, one can write for aggregate demand

$$(4) \qquad \theta(Y;\mu,M,A) := \begin{cases} G(Y;M) + \mu H(Y) + A, & 0 \leq Y < Y^o \\ G(Y;M) + A, & Y^o \leq Y \leq Y^F \end{cases}$$

Given this situation the chaos condition is simplified. By making μ large enough Y^o becomes a (local) minimizer and $Y_\mu^{min} = A + G(Y^o)$ so that $Y_\mu^* = Y^o$. Because of assumption A, $Y^C > Y^o > A + G(Y^o)$ and because the IY function can be stretched upward as much as we like by increasing μ there always exists some μ such that $\theta(A + G(Y^o)) \geqq Y^C$. Note also that, since G is monotonic it has an inverse. Consequently, $Y^C = G^{-1}(Y^o - A))$. If this value is less than Y^F then all oscillations are bounded below full employment.

To summarize we have

PROPOSITION II: If there exists CY and IY functions $G(\cdot)$ and $H(\cdot)$ satisfying assumptions A-D then there exist μ such that

$$\text{(KK1)} \qquad A + G[A + G(Y^o)] + \mu H[A + G(Y^o)] \geqq G^{-1}(A - Y^o) \ .$$

and if among these μ there exist some such that

$$\text{(KK2)} \qquad Y^F \geqq A + G[A + G(Y^o) + \mu H[A + G(Y^o)]$$

then all of the results of proposition I hold.

The Strict Keynesian Model

It is easy to show that Keynes' special assumptions about the demand for money and goods can lead to chaos. Recall first that the the transactions demand for money is given by $M_1 = kY$, where k is one over the transactions velocity of money. The demand for liquidity is given by the liquidity preference function $M_2 = L(i)$, $i \geqq i^* \geqq 0$. It is assumed that $L(\cdot)$ is monotonically downward sloping and so possesses an inverse for all $i \geqq i^*$ which is denoted by $L^{-1}(\cdot)$. The liquidity trap is expressed by the statement that $L(i)$ grows indefinitely large as interest falls to the level i^*, i.e., $L(i) \to \infty$ as $i \to i^{*}+$. Corre-

spondingly, $L^{-1}(M_2) \to i^*$ as $M_2 \to \infty$. We assume also that $L^{-1}(M_2) \to \infty$ as $M_2 \to 0$. These several assumptions give the profile for the liquidity preference curve as it is traditionally drawn .

Using the money demand functions $M^d = M_1 + M_2 = kY + L(i)$, assuming a fixed money supply $M^s = M$ and assuming a temporary monetary equilibrium $M^d = M^s$, we have the LM curve

$$(5) \qquad i = L^{-1}(M - kY) , \qquad 0 \leq Y \leq M/k .$$

Elementary application of the chain rule shows that this is a monotonically upward sloping function of national income Y. As Y approaches M/k the transactions demand increasingly absorbs the money supply, and, given the usual assumptions, the interest rate grows indefinitely. $Y^M = M/k$ thus forms a bound on national income. When Y is zero all money is available for liquidity and interest falls to a level consistent with the money supply, namely, $i' = L^{-1}(M)$.

Assuming a fixed marginal propensity to consume β the CY function is

$$(6) \qquad G(Y) = \beta Y , \qquad \text{with } 0 < \beta < 1 .$$

Finally, induced investment is assumed to be a monotonically downward sloping function of the interest rate $I = \mu I(i)$, which does not depend directly on income Y and where μ is interpreted as a parameter measuring the importance or intensity of induced investment demand, as before. Substituting (5) into this function we have the IY function

$$(7) \qquad I = H(Y) = \mu I[L^{-1}(M-kY)] .$$

Given (6) Assumption A is satisfied. Hence, for $\mu = 0$ we have the elementary Kahn-Keynes multiplier process. As μ increases, however, induced investment plays a role. We know from our preceding analysis that the behavioral implications of this role depend on the "shape" of the IY function. Again, the chain rule is used to determine that the investment income function is downward sloping. The lowest interest rate possible is $i' = L^{-1}(M)$, so investment is at a maximum at $H[L^{-1}(M)]$. Due to the crowding out effect on interest caused by the transactions demand for money induced investment falls at increasing levels of national income. Therefore, assumption B is satisfied. If induced investment is <u>sufficiently sensitive at high interest rates</u> so that investment falls rapidly enough at high income levels Condition K of Proposition I will hold. If in addition there exists a rate, i" say, at which investment becomes zero, and if the money supply is small enough so that interest rates rise very fast as Y approaches $\bar{Y} = \min\{M/k, Y^F\}$, then we have Macro Chaos Conditions KK1 and KK2 of Proposition II.

To summarize, chaos is a possibility inherent in the dynamic, fixprice model and in particular the Keynesian version of it. Examples of the sufficient conditions for models with specific functional forms are given in Appendix A.

3. DYNAMIC SCRAMBLING IN A KINKY KEYNESIAN WORLD

A Piecewise Linear Model

It is the nonlinearity of the macro model that produces the scrambling effect of dynamic adjustments. Nonetheless, if we linearize <u>but at</u>

the same time introduce sensible nonnegativity constraints we obtain some of the simplicity of the linear world but the scrambling of the nonlinear world. We therefore lose nothing essential and gain insight into the mechanism by which order generates disorder. Moreover, we can obtain a precise characterization of the stochastic-like character of chaotic behavior.

Putting in the Kinks

If we linearize and if the Keynesian Equilibrium is locally unstable, then national income, liquidity preference and investment demand must eventually become negative. In order to prevent these unmeaningful occurrences nonnegativity restrictions may be introduced. Thus, let the liquidity preference schedule be

$$(8) \qquad L(i) = \begin{cases} L^\circ - \alpha i \ , & 0 < i < L^\circ/\alpha \\ 0 \ , & L^\circ/\alpha < i \ . \end{cases}$$

(We have let the liquidity trap occur at zero though interest could be bounded below by a positive number just as well.) The implied LM curve is

$$(9) \qquad i = \begin{cases} 0 \ , & 0 \leqq Y \leqq Y^1 \\ (L^\circ - M + kY)/\alpha \ , & Y^1 \leqq Y \leqq M/k \ . \end{cases}$$

where $Y^1 = (M - L^\circ)/k$. Here it is assumed that $L^\circ < M$. (See Figure 2a)

The kinked investment demand function implied by the nonnegativity constraint is

$$(10) \qquad I(i) = \begin{cases} I^\circ - \gamma i \ , & 0 \leq i \leq I^\circ/\gamma \\ \\ 0 \ , & I^\circ/\gamma \leq i \ . \end{cases}$$

Because of the kinks, substituting (9) into (10) requires a bit of care. There are various possibilities depending on the relative positions of I° and L°. Suppose, to take one of these, that $I^\circ/\gamma < L^\circ/\alpha$. The maximum value of investment is I° which occurs when $i = 0$, that is, when income lies between zero and $Y^1 = (M-L^\circ)/k$. As income rises above the latter level, induced investment falls until it reaches its nonnegativity bound at I°/γ. Setting this equal to the interest rate from the LM curve, we get the second kink in the IY schedule. It occurs at that income where investment becomes zero, i.e., where $Y^\circ = Y^1 + \alpha I^\circ/k$. This gives the piecewise linear IY function

$$(11) \qquad H(Y) := \begin{cases} I^\circ & , \quad 0 \leq Y \leq Y^1 \\ I^\circ - \gamma(L^\circ - M + kY)/\alpha \ , & \quad Y^1 \leq Y \leq Y^\circ \\ 0 & , \quad Y^\circ \leq Y \leq M/k \ . \end{cases}$$

See Figures 2b and 2c.

The aggregate demand curve is now made up of three segments corresponding to the three linear pieces of the IY function. Setting $I^A = (\gamma/\alpha)(M-L^\circ)$ the dynamic equation (3) becomes

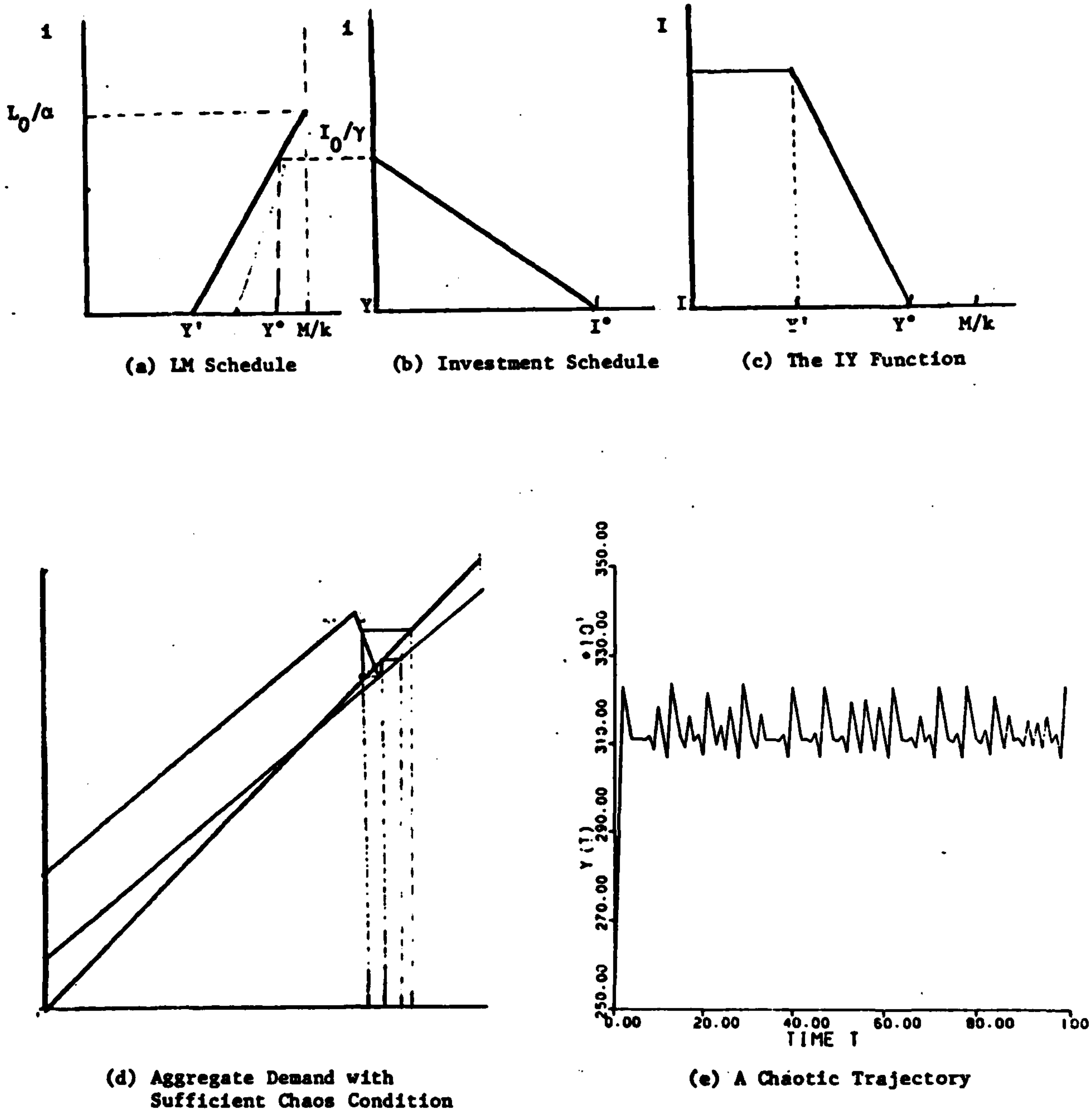


FIGURE 2: The Piecewise-Linear Keynesian
Model with Nonnegativity Kinks

$$(12) \qquad Y_{t+1} = \theta(Y_t) = \begin{cases} \beta Y_t + I^{\circ} + A & , \quad 0 \leq Y \leq Y^1 \\ [\beta - \gamma k/\alpha]Y_t + I^{\circ} + I^A + A \ , & Y^1 \leq Y \leq Y^{\circ} \\ \beta Y_t + A & , \quad Y^{\circ} \leq Y \leq M/k \end{cases}$$

When the middle segment is steep enough we get a diagram (see Figure 2d) that looks like a piece-wise linear approximation to the "general model" illustrated in Figure 1c.

Chaos Parameters

The middle segment has a negative slope when $\beta - (\gamma k/\alpha) < 0$. This is when cycles occur and when Y° becomes a local minimum of the aggregate demand function. Its preimage is $Y^c = (Y^{\circ}-A)/\beta$ which lies on the third segment of the aggregate demand curve. The minimum value also lies on the third segment, $Y^{min} = \beta Y^{\circ} + A$. Its "successor" is $\theta(Y^{min}) = \min\{\beta Y^{min} + I^{\circ} + A, [\beta-(\gamma k/\alpha)Y^{min} + I^A + A\}$. From Proposition II the parameter combinations leading to chaos are therefore found to be those that satisfy the inequalities

$$(13) \qquad \min\{M/k, Y^F\} \geq \min\{\beta(\beta Y^{\circ}+A) + A + I^{\circ}, \\ [\beta - \gamma k/\alpha](\beta Y^{\circ}+A) + I^A + A\} \\ \geq (Y^{\circ}-A)\beta.$$

A Closer Look at Dynamic Scrambling

The present example affords a particularly convenient setting in which to investigate the process of dynamic scrambling. What we want is an intuitive feel for how nonlinear feedback can cause erratic behavior.

We notice first of all that income trajectories must eventually be bounded below by Y^{min}. Thus at low initial levels income grows until this level is exceeded. After Y^1 is reached an overshoot of the Keynesian equilibrium, Y^k, occurs and income contracts. It must fall below Y^k after one or a few periods but it cannot fall below Y^{min}. This also means that Y_t is eventually bounded above either by $Y^{max} = \theta(Y') = \beta(M-L^\circ)/k + I^\circ + A$, if $Y^{min} \leqq Y'$ or by $Y'' = \theta(Y^{min}) = (\beta-\gamma k/\alpha)(\beta Y^\circ+A) + I^\circ + I^A + A$ if $Y' \leqq Y^{min}$. Let

$$(14) \qquad Y^m = \begin{cases} Y^{max} & \text{if} \quad Y^{min} \leqq Y' \\ Y'' & \text{if} \quad Y' \leqq Y^{min} . \end{cases}$$

Then all the action of the model eventually takes place in the interval $[Y^{min}, Y^m]$.

By defining the linear transformation $y = (Y-Y^{min})/(Y^m-Y^{min})$ we get instead of (12) a kinked dynamic process in y_t

$$(15) \qquad y_{t+1} = T(y_t) := \begin{cases} a^1 + \beta y_t & , \quad 0 \leqq y \leqq y^1 \\ a^2 + [\beta - \gamma k/\alpha]y_t, & y^1 \leqq y \leqq y^\circ \\ a^3 + \beta y_t & , \quad y^\circ \leqq y \leqq 1 \end{cases}$$

where

$$a^1 = \frac{I^\circ + A + \beta Y^{min}}{Y^m - Y^{min}} \quad , \quad y^1 = \frac{(M - L^\circ)/k - Y^{min}}{Y^m - Y^{min}}$$

$$a^2 = \frac{I^\circ + I^A + A + \beta Y^{min}}{Y^m - Y^{min}} \quad , \quad y^\circ = \frac{Y^\circ - Y^{min}}{Y^m - Y^{min}}$$

$$a^3 = \frac{A + \beta Y^{min}}{Y^m - Y^{min}} \quad .$$

This process has exactly as many kinks as (12) and exactly the same dynamic behavior. However, instead of bounded oscillations in S we get bounded oscillations in [0, 1].

There are two distinct forms the map (15) can take, one when $Y^{min} \leq Y'$ and one when $Y^{min} \geq Y'$. These are shown in Figure 3. For simplicity we are going to confine attention here to the second one of these shown in Figure 3b. For this map we have $y^{min} = 0$ and $y^{max} = 1$. The first regime does not arise in [0, 1]. On the unit interval, therefore, we can simplify (15) to obtain

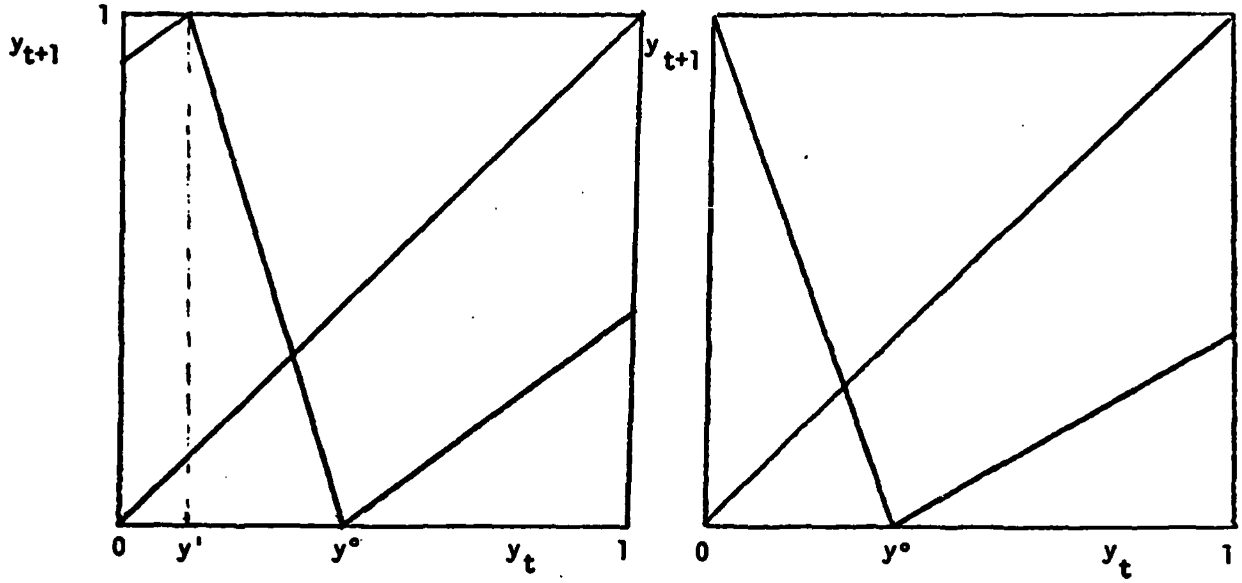
$$(16) \qquad y_{t+1} = T(y_t) := \begin{cases} 1 + [\beta - \gamma k/\alpha]y_t & , \quad 0 \leq y \leq y^\circ \\ -\beta y^\circ + \beta y & , \quad y^\circ \leq y \leq 1 . \end{cases}$$

with $y^\circ = 1/(\gamma k/\alpha - \beta)$.

As a final simplification, consider the case where a three period cycle exists with periodic points 0, y°, 1. This case is illustrated in Figure 3b. Now we have $y^c = 1$, $y^* = y^\circ$ and $y^{min} = 0$. The Proposition II conditions are therefore satisfied and chaotic trajectories exist.

We note in this special case that $\beta(1-y^\circ) = y^\circ$, so $y^\circ = \beta/(1+\beta)$. Consequently, our remaining parameters are not independent. Specifically, the relationship $1+2\beta/b = \gamma k/\alpha$ exists among them. Now $\beta/(1+\beta) < 1$ for all $\beta > 0$. Hence, $1/y^\circ > 1$. But $1/y^\circ = -(\beta-\gamma k/\alpha)$ so that $\beta-\gamma k/\alpha < -1$. This means that the Keynesian Equilibrium is unstable: no matter how close income comes to y^k it must move away in initially growing oscillations.

But where do income trajectories go? We know that they do not converge to a cycle of any order so they do not bunch up around periodic



(a) Two Kinks in [0,1]

(b) One Kink in [0,1]

FIGURE 3: Two Possible Maps on the Unit Interval for the Piecewise-Linear, Transformed Keynesian Model.

points. It is easy to see why this is so. Consider the sequence of maps $\theta^1(y)$, $\theta^2(y)$, $\theta^3(y)$,... where $\theta^n(y) = \theta(\theta^{n-1}(y))$ $n = 1,\ldots,$ with $\theta^0(y) := y$ and $\theta^1(y) := \theta(y)$. It is an extremely tedious job to compute the algebraic form of these functions because they involve an increasing number of kinks. Constructing the first several maps in the series, however, presents no great difficulties. This has been done in Figure 4. As one can see in diagrams a-d, when the number of iterations grows, the number of kinks grows, and with it the number of fixpoints, the latter representing successively higher order cycles. The piecewise linear segments of successive maps become steeper so that the cycles get less and less stable as their order increases. The same phenomonon also arises in the general example underlying Figure 1 and described in Appendix A. The first four iterates of this map are shown in Figure 5.

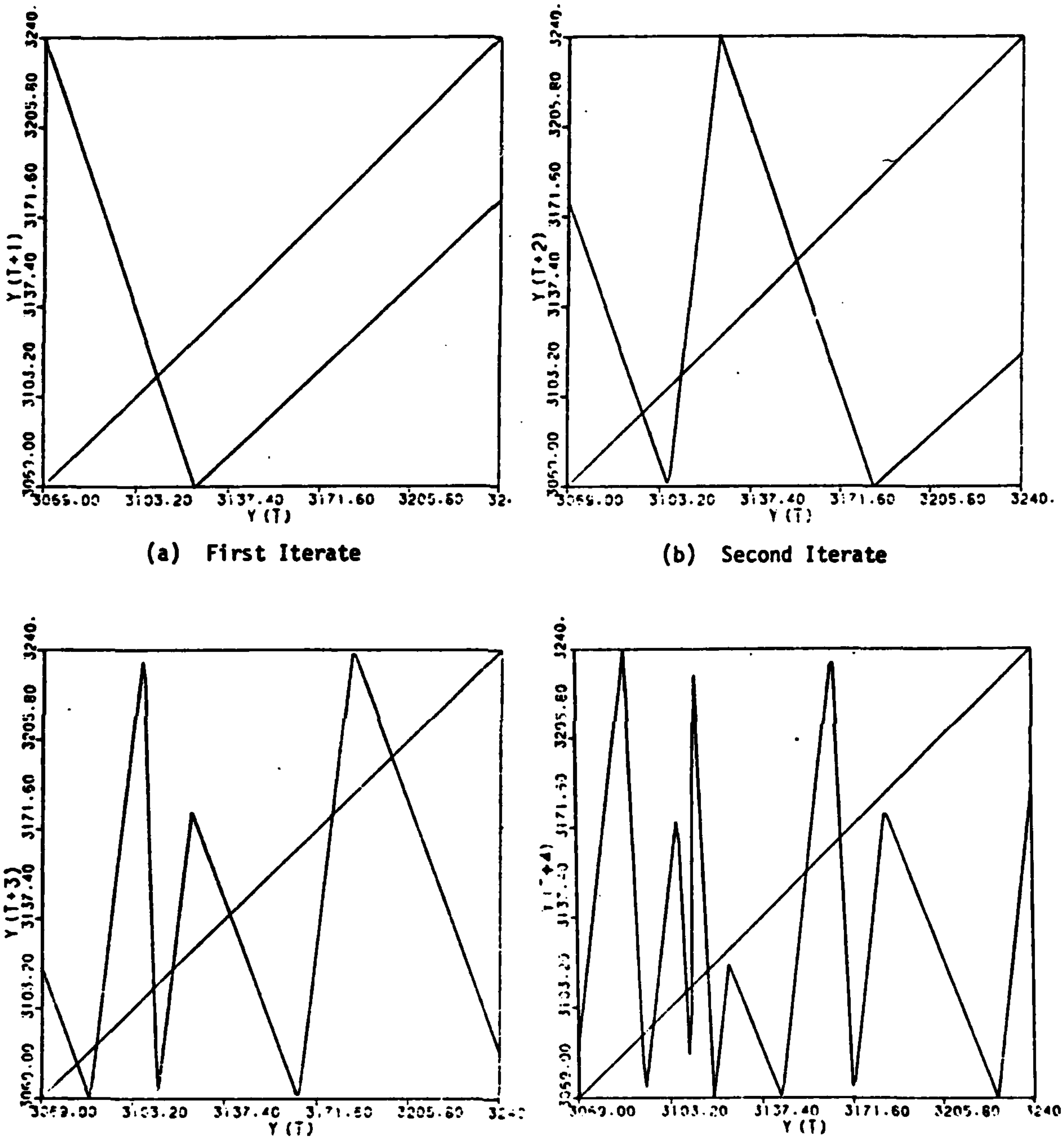
As one can see, more and more fixpoints get crowded into the zero-one interval. These are distributed throughout the interval. As a consequence nonperiodic trajectories always lie near a cycle of <u>some</u> order, but as these are unstable one always moves away from any given periodic cycle.

There are a countably infinite number of periodic points which cut the interval a countably infinite number of times. What remains contains the scrambled set of the Li-Yorke Theorem. All nonperiodic trajectories are attracted to this disconnected set and, since it is infinitely layered, so to speak, it is not inappropriate to call it "strange."

4. EROGODICITY

Some Background

If chaotic trajectories were rare in the sense that a random selection would lead to a small or zero probability of choosing an initial



(a) First Iterate

(b) Second Iterate

(c) Third Iterate

(d) Fourth Iterate

FIGURE 4: Iterates of the Piecewise-Linear Keynesian Model in the Chaos Range.

condition in the scrambled set, then they would be of much diminished importance. Mathematically one wants to know, does the scrambled set S have positive measure? Also, because they cannot be represented by finite, closed form solutions, one would like to know how to characterize nonperiodic trajectories. For example, are they essentially like stochastic processes? These questions cannot be answered directly at the present state of knowledge. However, some concepts from ergodic theory can be used to show that in some cases where chaotic trajectories in the sense of Li and Yorke exist, <u>almost all trajectories will be highly irregular</u>, taking on the character of stationary stochastic processes. First we will provide a brief discussion of the application of ergodic theory to the study of difference equations.

Let T be a continuous map of the unit interval $[0,1]$ into itself, and consider the difference equation $x_{t+1} = T(x_t)$. For any initial condition x_0, the solution can be written $x_t = T^t(x_0)$, $t = 0, 1, 2, \ldots$ where T^t is the t^{th} iterate of T. One way to describe the structure of the time path starting at x_0 is to compute, for each n, the cumulative frequency distribution of the time path up to time $n-1$. To do this we use the indicator function for a set S defined by

$$\Delta(x,S) = \begin{cases} 1, & x \in S \\ 0, & x \notin S . \end{cases}$$

Thus, define

$$(17) \qquad F^n(x,x_0) := \frac{1}{n}\sum_{t=0}^{n-1} \Delta(\theta^t(x_0),\ [0,x]),\ 0 \leq x \leq 1 \quad .$$

The function F^n gives the fraction of the n points x_0, $x_1 = \theta(x_0),\ldots,$ $x_{n-1} = \theta^{n-1}(x_0)$ that lie in the interval $[0,x]$.

For the system to be ergodic we mean that there exists a distribution function F such that, if one picks an initial condition x_0 at random from the unit interval according to the uniform distribution, then with probability one the empirical distribution functions $F^n(\cdot;x_0)$ will converge in distribution to F:

(18) $$\lim_{n\to\infty} F^n(x:x_0) = F(x) \quad \text{at each continuity point } x \text{ of } F, \text{ for almost all } x_0.$$

Whether or not the behavior of the time paths are nice or erratic is now completely characterized by F. For example, suppose T has a unique globally stable stationary point x^*. Then for every x_0, $F^n(\cdot;x_0)$ will converge to the F which concentrates all probability at x^*, i.e., $F(x) = 0$ for $x < x^*$ and $F(x) = 1$ for $x \geqq x^*$. Or consider the case where $T(x) = (.5x)^{.5}$. In this case there are two stationary points $x^* = 0$ and $x^{**} = .5$. It is not difficult to check that for any $x_0 > 0$ the time path converges to $x^{**} = .5$, so that for almost all x_0, $F^n(\cdot;x_0)$ converges to the F which concentrates all probability at .5. For a slightly more complicated example, consider $T(x) = 1-x^2$. There is one stationary point $x^* = (5^{.5}-1)/2$ and a two period cycle $T(0) = 1$, $T(1) = 0$. The stationary point is unstable but the cycle is an attractor: for any x_0 different from x^*, the time path will converge to the two cycles, so that $F^n(\cdot:x_0)$ converges to the F given by $F(x) = 0$ for $x < 0$, $F(x) = .5$ for $0 < x < 1$, and $F(x) = 1$ for $x > 1$.

In the above examples the behavior is very regular, a stable stationary point or cycle, and this is exactly reflected in F having

discontinuities or jumps which reflect concentrated probabilities at points. The opposite will be true if F is continuous. A sufficient condition for this is that F have a density f, i.e., $dF(x) = f(x)dx$. In this case, if (18) holds then no stationary point or cycle can be locally stable, and for almost all x_0, the resulting time path will spend an infinite amount of time arbitrarily close to almost any x such that $f(x) > 0$, and an infinite amount of time a finite distance away from x. Thus, the behavior will be extremely irregular; in fact the time paths will be indistinguishable from the realizations of a stationary ergodic stochastic process with common density f, and in this sense will appear random. All this from a deterministic and extremely simple model!

<u>The "Normalized, Keynesian Model."</u>

We now consider the normalized Keynesian model (16) in the last section. There T has the form

$$T(y) = \begin{cases} 1-y/B & \text{for } y < B \\ C(y-B)/(1-B) & \text{for } y \geq B , \end{cases} \tag{19}$$

where $0 < B := y^{\circ} < 1$ and $C = \beta(1-y^{\circ})$.

We show in the appendix that there exists a F with density f such that (18) holds, in the two cases $B(1-B) < C \leq B$ and $B^{.5}(1-B) < C \leq (B + (B^2 + 4B(1-B))^{.5})/2$. In particular, we are able to exhibit the density in the special case where $C = B$: in this case

$$(20) \qquad f(y) = \begin{cases} 1/(2-B) & \text{for } y > B \\ 1/B(2-B) & \text{for } y < B \end{cases}$$

Thus, for almost any initial condition y_0 the time path will asymptotically spend the fraction of time $1/(2-B)$ in the interval $[0,B]$, the fraction of time $(1-B)/(2-B)$ in the interval $[B,1]$, and will spread itself evenly over each interval. See Figure 6 for a simulation of this model and note how it compares to the theoretical results.

Ergodicity in the General Case.

It has proven extremely difficult to establish conditions for ergodicity for general, nonlinear models such as the general fixprice macro economy of equation (3) even when chaos is known to exist. However, because the piece wise linear, Keynesian model is an approximation of the more general case it seems possible that ergodicity does hold for less tractable examples. Numerical experiments add credence to this supposition. Using the quite general functional forms elaborated in Appendix A and that underly the chaotic trajectory illustrated in Figure 1d, we compute the cumulative frequency distribution shown in Figure 7b, which appears to be quite stable with respect to changes in the initial condition.

Although, we would expect that more general functions would imply more complex behavior than the simple piecewise linear case it is perhaps a shock to see that the implied density function, shown in Figure 7a, is a highly irregular object!

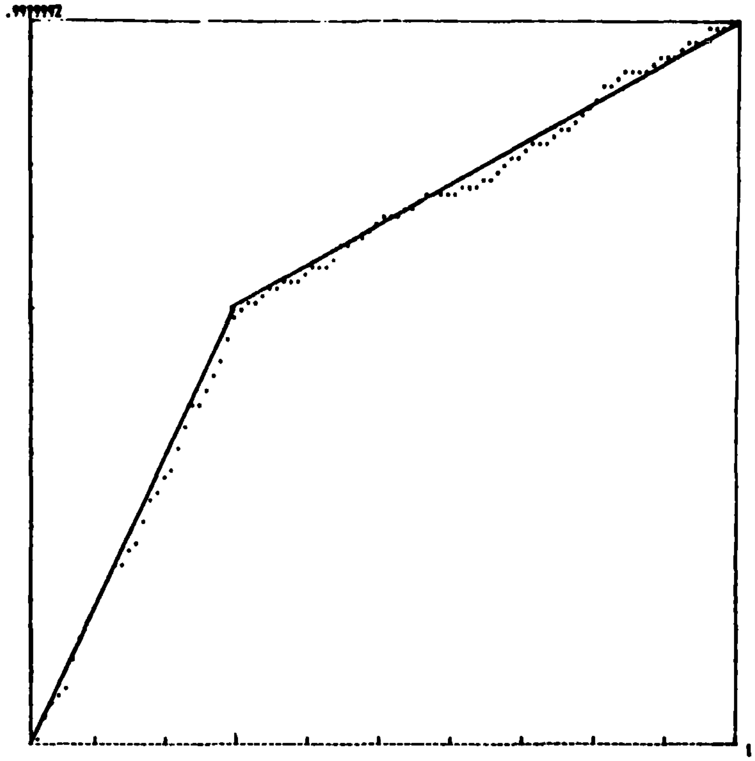
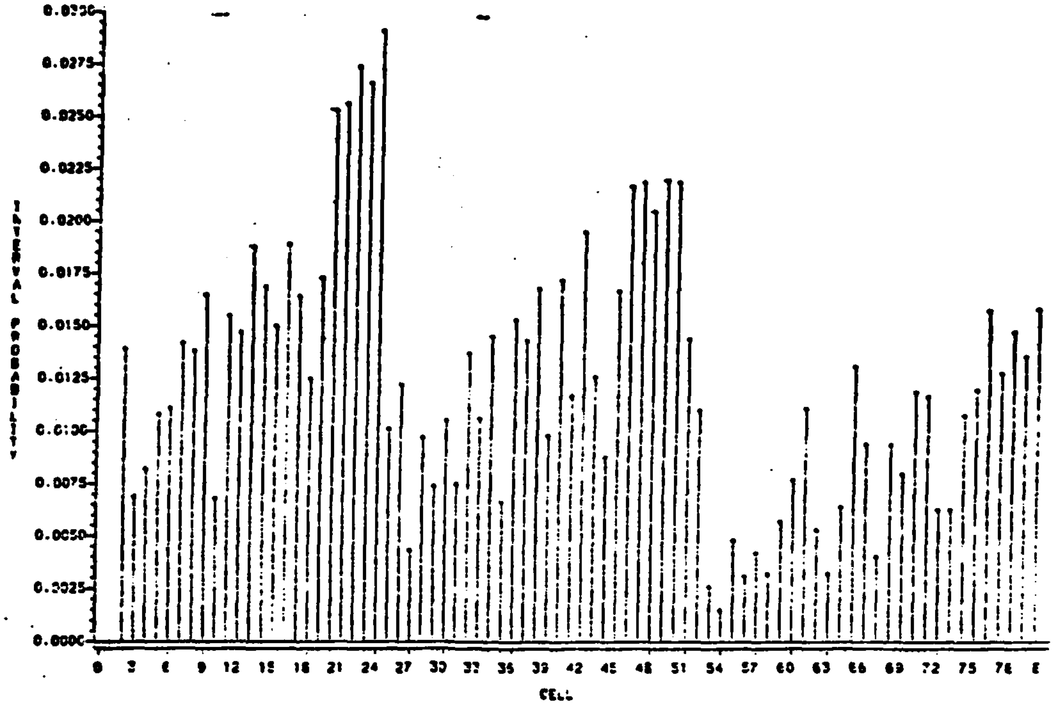
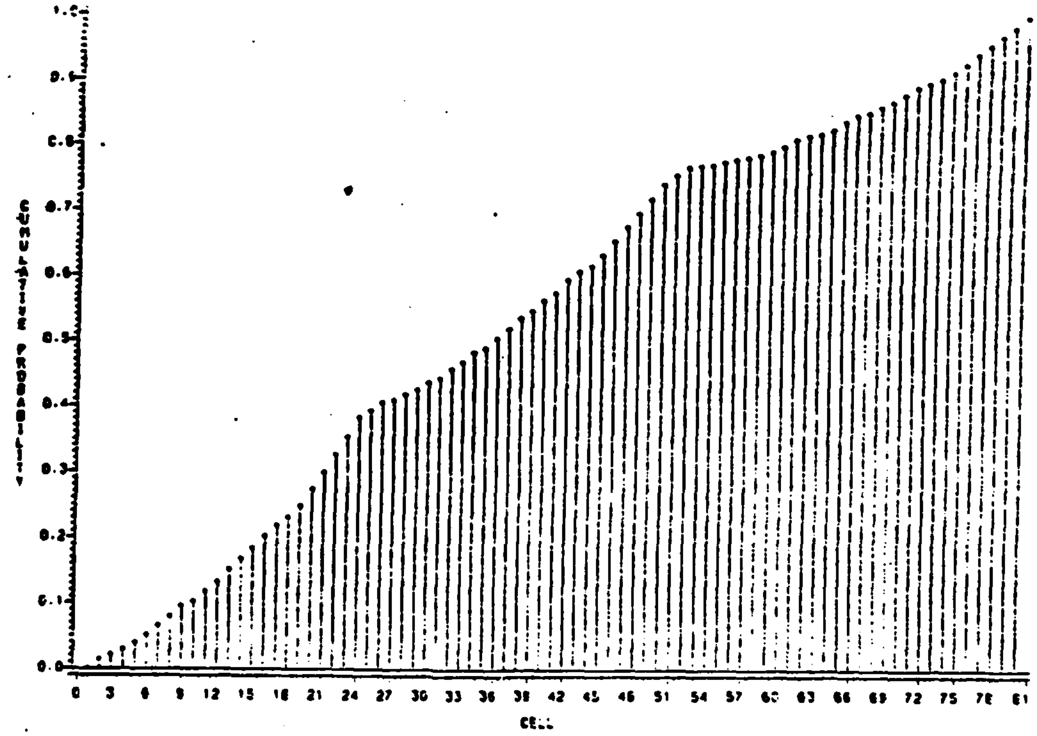


FIGURE 6: Theoretical and Simulated Cumulative Distribution Function for the Piecewise-Linear Keynesian Model.

PARAMETERS B = .3 ,C = .3 ,N = 200 ,AND INITIAL VALUE .2162.



(a) Simulated Density Function



(b) Simulated Cumulative Distribution Function

FIGURE 7: "Ergotic" Property of "General" Fixprice Model

5. MONEY AND STABILITY

Let us conclude by considering how macroeconomic behavior is influenced by changes in the money supply within the fixprice framework. As with the study of ergodicity this is most easily done for the piecewise linear model (8)-(12). Three regimes can be identified depending on which branch of the aggregate demand curve "contains" the Keynesian equilibrium, Y^k.

Thus, if $0 \leq Y^k \leq Y^1$ we have from (12) $Y^k = (A+I^\circ)/(1-\beta)$ which is a stable high income, low unemployment equilibrium. To obtain this case the money supply must be great enough so that $Y^k = (A+I^\circ)/(1-\beta) \leq Y^1 = M/K - L^\circ/K$ or $M \geq M^u = K(A+I^\circ)/(1-\beta) + L^\circ$. Let us call this Regime I. In it the money supply is so generous that the interest rate is at its lower bound (the liquidity trap prevails) and induced investment is at its upper bound.

Or, suppose that $Y^\circ \leq Y^k \leq M/K$. Then, again from (12) we have $Y^k = A/(1-\beta)$ which is a stable, low income, high unemployment equilibrium. The money supply is now relatively tight and interest rates are so high that induced investment has been crowded out completely. Call this Regime III. To get it we must have $Y^\circ = (M-L^\circ)/K + \alpha I^\circ/K \leq A/(1-\beta)$ or $M \leq M^\ell = kA/(1-\beta) + L^\circ - \alpha I^\circ$. Here, because of the crowding out effect, equilibrium income depends exclusively on autonomous expenditure.

Let us call Regime II the case in which $M^\ell \leq M \leq M^u$. In this regime induced investment is positive, and if Y^k is unstable, there may exist money supplies for which chaotic cycles exist. Suppose the sufficient conditions (13) for this possibility do exist. Then various interesting and not wholly implausible scenarios for monetary policy might be constructed.

For example, suppose the money supply is initially compatible with Regime II, and irregular investment, interest and income cycles emerge. A sufficient reduction in the money supply would push the economy into Regime III, stabilizing macrobehavior which would approach a low income level, high unemployment, equilibrium. Contrastingly, if the money supply were increased enough, Regime I would come into play with convergence to a high income level, low unemployment Keynesian equilibrium. Monetary policy would have two alternative routes to stabilization, and two stable regimes.

Or, suppose the economy were converging on a low income Keynesian equilibrium in Regime III. A big enough increase in the money supply could now <u>de</u>stabilize the economy and induce chaotic oscillations in interest, investment and income. One imagines that the monetary authorities might be frightened away from a liberal monetary policy even though by pushing it to greater extremes a stable equilibrium at low interest rates could be induced.

Finally, suppose an equilibrium in Regime I is being approached but, due to a shift in basic policy a tight money policy is introduced, enough so that Regime II is entered. Again, chaotic cycles can emerge. In this case one might imagine the frightened authorities reverting to a softer monetary approach after a relatively brief experience with chaos.

These rather striking departures from the usual comparative static analysis of the Keynesian system give tantalizing hints of important new insights into monetary theory and policy when comparative dynamics are studied in the presence of nonlinearities of the kind illustrated in this paper.

APPENDIX A

In this appendix we derive the sufficient chaos conditions for a nonlinear Keynesian model with specific functional forms for liquidity preference and investment demand.

<u>Example 1: Nonlinear Investment and Liquidity Preference Functions</u>

For the liquidity preference schedule take the function $M_2 = \lambda/(i-i^*)$, $i \geqq i^*$. Its inverse is $i = i^* + \lambda/M_2$ so that the LM curve is

$$\text{(A1)} \qquad i = i^* + \lambda/(M-kY) \ , \qquad Y \leqq M/k \ .$$

This has the usual Keynesian shape.

Consider as an investment schedule the log-linear function

$$\text{(A2)} \qquad I(i) = \begin{cases} \left(\frac{i''-i}{i}\right)^b , & 0 \leqq i \leqq i'' \\ 0 & i'' \leqq i \end{cases}$$

It has the property that investment falls to zero when interest approaches i'' and rises as rates fall from this level. For simplicity in what follows we set $b = 1$.

To get the IY function we need the income level Y^o at which interest rates reach i''. To do this use the LM curve. We get $Y^o = \frac{1}{k}[M - \frac{\lambda}{i''-i^*}]$. Hence, we have the IY function with two regimes

$$\text{(A3)} \qquad H(Y) := \begin{cases} i''/\left(i^* + \frac{\lambda}{M-kY}\right) - 1, & 0 \leq Y \leq Y^\circ \\ 0 \qquad , & Y^\circ \leq Y . \end{cases}$$

All the requirements of Assumptions A-D are satisfied. Consequently, Proposition II applies. Using it we find with the help of a little routine algebra, that the intensity of induced investment sufficient for chaos to emerge are values of μ satisfying

$$\text{(A4)} \qquad \frac{\bar{Y} - \beta^2 Y^\circ - (1+\beta)A}{i''/\left[i^* + \frac{\lambda}{M-k(A+\beta Y^\circ)}\right] - 1} \leq \mu \leq \frac{(1+\beta^2)Y^\circ - (1+\beta+\beta^2)A}{i''/\left[i^* + \frac{\lambda}{M-k(A+\beta Y^\circ}\right] - 1}$$

We know from our analysis leading to Proposition II that intensities of investment satisfying the right side always exist. But for feasible oscillations to exist, $\bar{Y}$ must be large enough. This can happen only when there is a Keynesian unemployment "equilibrium" sufficiently below Y^M and Y^F.

Example 2: A "Kaldorian" Investment Function

To illustrate a "general" fixprice model consider a Kaldorian type investment function for which low levels of income depress investment and high levels stimulate it. Thus, replace (A2) with

$$\text{(A5)} \qquad I(i,Y) := Y^\alpha(\bar{\bar{i}}-i)^\delta$$

The IY function now becomes

(A6) $$H(Y) := Y^{\gamma}\left(\frac{[(\bar{\bar{i}}-i^*)(M-kY) - \lambda]}{M - kY}\right)^{\delta}$$

The diagrams of Figure 1 are based on this example and the trajectory shown in 1d was generated using it. The results shown in Figure 7 suggest that for the parameters chosen the model may be ergodic, but with a stochastic structure that is itself extremely complex!

APPENDIX B

This appendix presupposes a basic knowledge of measure theory. The ergodic theorems underlying what follows can be found in Dunford and Schwartz [1957], Cornfeld et al. [1981], and Parthasarathy [1967]. Let P denote Lebesgue measure on the Borel subsets of [0,1], and T a continuous map of [0,1] into itself. A sufficient condition for (18) to hold for some F absolutely continuous with respect to P, or equivalently for the probability measure Q corresponding to F, is that:

i) There exists a constant M such that $P(T^{-k}(A)) \leq MP(A)$ for all $k > 0$ and all Borel sets A.

ii) If A is any Borel set and $T^{-1}(A) = A$ then $P(A) = 1$ or 0. If i) and ii) are satisfied the invariant measure Q corresponding to F in (18) will be the unique measure absolutely continuous with respect to P satisfying $Q(T^{-1}(A)) = Q(A)$ for each Borel set A. In order to verify i) there is fortunately a result in the literature due to Kosyakin and Sandler [1972], and Lasota and Yorke [1973] which applies in our case. This theorem, specialized to our case of a piecewise linear T, states that i) will be satisfied if there exists an integer $k > 0$ and a number $z > 1$ such that $|T^{k'}(x)| \geq z$ for all x for which the derivative is well defined. In order to verify ii) there seems to be no general results which are applicable and we shall have to use brute force.

<u>Verification of i)</u>. First we do the case where $C \leq B$ and $C > B(1-B)$. In this case we show that $|T^{2'}(x)| \geq C/B(1-B)$. For any x, $T^{2'}(x) = T'(T(x))T'(x)$. Note that $|T'(x)| = 1/B > 1$ if x is in [0,B] and

$T'(x) = C/(1-B) < 1$ if x is in $(B,1]$. Since $C \leq B$, both x and $T(x)$ cannot lie in the interval $(B,1]$, so $|T^{2'}(x)| \geq C/B(1-B) > 1$. For the case where $C^2 > (1-B)^2B$ and $C \leq (B + (B^2 + 4B(1-B))^{.5})/2$ we will show that $|T^{3'}(x)| \geq C^2/B(1-B)^2$. Now $T^{3'}(x) = T'(T^2(x))T'(T(x))T'(x)$, so it suffices to show that x, $T(x)$, and $T^2(x)$ cannot all lie in the interval $(B,1]$. This will be satisfied if $T^2(1) \leq B$ or equivalently $T(C) = C(C-B)/(1-B) \leq B$, which corresponds to the upper limit on C in this case.

Verification of ii). Define $g:[0,B] \to [0,1]$ by $g(x) = 1-x/B$ and $h:[B,1] \to [0,C]$ by $h(x) = C(x-B)/(1-B)$. Note that, for any Borel set A, $T^{-1}(A) = g^{-1}(A) \cup h^{-1}(A \cap [0,C])$, and $P(g^{-1}(A)) = BP(A)$ and $P(h^{-1}(A \cap [0,C])) = C^{-1}(1-B)P(A \cap [0,C]$. Now consider any A such that $T^{-1}(A) = A$. Then:

$$
\begin{aligned}
P(A \cap [0,x]) &= P(T^{-1}(A) \cap [0,x]) \\
&= P(g^{-1}(A) \cap [0,x]) + P(h^{-1}(A \cap [0,C]) \cap [0,x]) \\
&= BP(A \cap [g(\min(x,B),1]) \\
&\quad + C^{-1}(1-B)P(A \cap [0,h(\max(x,B)])
\end{aligned}
$$

We will show that $P(A \cap [0,x]) = xP(A) = P([0,x])P(A)$ for all x. From this it is straightforward to show that $P(A \cap D) = P(D)P(A)$ for D any finite union of intervals and thus by a standard argument that $P(A \cap A) = P(A)P(A)$, so that $P(A) = 1$ or 0. To show this, we do the following steps:

a) We show it holds for $x = B$.

b) We show that if it holds for x, and $T(y) = x$, then it holds for y.

c) We show that $\cup_{k>0} T^{-k}(B)$ is dense in $[0,1]$.

d) Since $P(A \cap [0,x])$ is continuous in x and a), b), and c) imply it holds on a dense set, it holds for all x.

<u>Step a)</u>: From the formula for $P(A \cap [0,x])$ we get $P(A \cap [0,B]) = BP(A \cap [0,1]) + C^{-1}(1-B)P(A \cap [0,0]) = BP(A)$.

<u>Step b)</u>: Suppose it holds for x, and $T(y) = x$. If y is in $[0,B)$ then $y = B(1-x)$ and if y is in $(B,1]$ then $y = B+C^{-1}(1-B)x$. For y in $[0,B)$;

$$\begin{aligned} P(A \cap [0,y]) &= BP(A \cap [x,1]) + C^{-1}(1-B)P(A \cap [0,0]) \\ &= B(P(A) - P(A \cap [0,x))) = B(1-x)P(A) = yP(A) \end{aligned}$$

For y in $(B,1]$;

$$\begin{aligned} P(A \cap [0,y]) &= BP(A \cap [0,1]) + C^{-1}(1-B)P(A \cap [0,x]) \\ &= (B+C^{-1}(1-B)x)P(A) = yP(A) \end{aligned}$$

<u>Step c)</u>: It suffices to show that for any interval (a,b), B is contained in $T^n((a,b))$ for some $n > 0$. Suppose not. Then for every n, $T^n((a,b))$ is the image of $T^{n-1}(a,b)$ under g or h and thus is contained in $[0,B)$ or $(B,1]$. Decompose $T^n((a,b))$ into a sequence of compositions of g and h: and consider the number of times h appears. In the case $C \leq B$, h maps $(B,1]$ into $(0,B]$, so each h must be followed by a g. Thus if n is even, $P(T^n((a,b))) \geq |g'h'|^{.5n}P((a,b))$. Since $|g'h'| > 1$ in this case we would get $P(T^n((a,b))) > 1$ for n large enough, a contradiction. In the second case we are considering, h cannot appear more than twice in succession, so if $n = 3m$ for some m then $P(T^n((a,b))) \geq (|g'|(h')^2)^m P((a,b))$, and we get a similar contradiction.

<u>Step d)</u>: Self evident.

Finally we promised the exact formula for the density of Q in the case $C = B$. The formula given earlier can be written as $Q(A) = (1/B(2-B))P(A \cap [0,B] + (1/(2-B))P(A \cap [B,1])$, and all that is required is to show that $Q(T^{-1}(A)) = Q(A)$. This is straightforward using the decomposition $T^{-1}(A) = g^{-1}(A) \cup h^{-1}(A \cap [0,B])$, and is left to the reader.

NOTES

1. Burns and Mitchell for example observed that "...Every realistic investigation recognizes that business activity at any time is influenced by countless 'random' factors... . Each specific ... business cycle is therefore an individual differing in countless ways from every other." (Burns and Mitchell, pp. 446-467)

Marschak, in his introductory chapter to the definitive Cowles Monograph No. 14, set forth a similar view: "Exact structural relations ... are ... unrealistic. ...an unexplained residual would remain. It is called 'disturbance' or 'shock', and can be regarded as the joint effect of numerous separately insignificant variables that we are unable or unwilling to specify... . Shock and errors can be regarded as random variables." (Marschak, p. 12)

2. For a review of some of the relevant literature which is becoming voluminous see Day [1982] and Yorke and Yorke [forthcoming].

3. Benhabib and Day initiated this approach in studies of experience dependence and of exchange and accumulation in overlapping generations models (1981, 1982). Stutzer gives a thorough analysis of Haavelmo's growth model. In the meantime analysis of price adjustment (Montrucchio), Kaldorian Cycles (Malgrange and Dana) and Goodwin Cycles (Phojola) have appeared.

4. Since Samuelson's original analysis (1948, pp. 281-283) various authors have investigated dynamic versions of the Keynesian system, for

example Peacock [1962] and Smyth [1974]. Some of these linearize the consumption, investment and demand for money functions. As they also ignore nonnegativity restrictions, endogenous chaos, which is induced by nonlinearities, is eliminated. Other authors, like Samuelson, are concerned exclusively with local stability. They exploit linearizations of the structural relationships in the neighborhood of equilibria, again, with the effect of precluding an analysis of endogenous chaos. Pohjohla [1982] introduces a progressive linear tax function into the linear Peacock and Smyth models. This creates a nonlinearity in the consumption function leading to chaos result. Here we are concerned, however, with global results in which the phenomenon of interest is shown to depend not on some kind of government interference but on precisely the nonlinear properties Keynes thought were intrinsic to the demand for money and investment goods.

Perhaps it should be noted that intrinsic unpredictability does not preclude the possibility of random shocks. In econometric work we might indeed find it suitable to recognize unexplained, more or less random influences. Here, however, we do not wish to confound these two sources of irregularity and so assume that all forces are endogenous and purely deterministic.

5. For a recent discussion of the dynamic multiplier see Hicks, pp. 13ff.

6. When condition (L-Y) occurs there exist cycles of every order in J; i.e., for each n there exist at least n cyclic points satisfying $x = \theta^n(x)$, $n = 1, 2, 3, \ldots$; moreover, there exists an uncountable, scrambled

set $S \subset J$ with the following properties: all trajectories with initial conditions in S remain in S; every trajectory in S wanders arbitrarily close to every point in S infinitely often (trajectories are dense in S); no matter how close two distinct trajectories in S come to each other they eventually wander away; (instability of chaotic trajectories) and every trajectory in S wanders away from any cyclic trajectory in J, however close it may approximate it for a time (non-periodicity). A stronger and simpler criterion has been produced in Li, Misiurewicz, Pianigiani and Yorke [1982] which could also be applied in the present study. For purposes of tracing the existence of chaos to underlying economic structure the original Li-Yorke condition that we use here is perhaps more transparent.

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This paper shows how chaotic fluctuations can emerge in a dynamic version of the standard fixprice macroeconomic model. The crucial parameters involve induced investment and its sensitivity to interest rates. We use some ideas from the theory of chaos and from ergodic theory to illustrate the fact that apparently random trajectories need not be rare. The paper (cont.)

ABSTRACT (cont.)

concludes with some tantalizing hints at how insights on monetary theory and policy might change when comparative dynamics rather than comparative statics are used and when nonlinearities of the kind illustrated in the paper prevail.

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